Indian Statistical Institute
B. Math (Hons.) second year
Computer Science II- Numerical Methods
June 15, 2018

Back paper exam Time: 3 hours 50 points

1. [10 points] Let $x_0, x_1, \ldots, x_n \in [a,b]$ are distinct and $f \in C^{n+1}[a,b]$. Write down Lagrange interpolating polynomial $L_n(x)$ that satisfies $L_n(x_i) = f(x_i)$ for $i = 0, 1, \ldots, n$. Show that
$$f(x) - L_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} w(x)$$
where c depends on x, $f^{(n+1)}$ denotes the $(n+1)$th derivative of f and $w(x) = \prod_{k=0}^{n}(x - x_k)$. If the points $x_0, x_1, \ldots, x_n$ are equally spaced given by $x_0 = a, h = \frac{b-a}{n}, x_j = a + jh$, show that
$$|f(x) - L_n(x)| < \frac{h^{n+1}}{4(n+1)} \max_{c \in (a,b)} |f^{(n+1)}(c)|.$$

2. [5 points] Define `flops`. Estimate upto 'Big O' notation the flops required for calculating determinant of a matrix. Let $A = LU$ be the LU factorization where L is a lower triangular with $1's$ on its diagonal and U is an upper triangular. Determine the flops required to calculate determinant of A.

3. (a). [1 point] Define symmetric positive definite matrices.
   (b). [2 points] Let A be a symmetric positive definite matrix. Show that A is invertible and the diagonal elements of A are positive.

4. (a). [5 points] Write an Octave/Matlab function `linefit` to fit a line to a given data.
   (b). [2 points] Write an `xlinfit` function that calls `linefit` to fit data to $y = \frac{x}{c_1 x + c_2}$.

5. (a). [4 points] Establish Newton-Cotes formula for approximating $\int_a^b f(x)dx \approx \sum_{i=0}^{n} c_i f(x_i)$ where f is a continous function defined on $[a,b]$ and its values are known at the $n+1$ equidistant points $a = x_0 < x_1 < \ldots < x_n = b$.
   (b). [2 points] Show that $\sum_{i=0}^{n} c_i = 1$.
   (c). [4 points] Deduce Trapezoidal and Simpson's $\frac{1}{3}$ rule from Newton Cotes formula.
   (d). [10 points] Write an Octave function `betatrap` to evaluate
$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$$
   for any m and n and for a sequence of decreasing panel sizes h using trapezoidal approximation. Comment on the error in evaluating $\beta(1,2)$.

6. (a). [2 points] Show that the following IVP
$$\frac{dy}{dt} = y \cos t, \ 0 \le t \le 1, \ y(0) = 1,$$
   has unique solution.
   (b). [3 points] Manually perform 3 steps of Euler method to solve
$$\frac{dy}{dt} = t - 2y, \ y(0) = 1$$
   with $h = 0.2$